JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **100**, 242–249 (1984)

On Oscillation of Second-Order "Sublinear" Differential Equations

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Submitted by R. P. Boas

1. INTRODUCTION

The behaviour of solutions, as oscillatory or non-oscillatory, of differential equations of the form

$$x''(t) + a(t)\, f(x) = 0, \qquad t \in [0, \infty),$$

where $a(t)$ and $f(x)$ are functions satisfying certain prescribed general conditions, have received the attention of many authors in recent years. A fairly extensive bibliography of the earlier works can be found, for instance, in the survey article by Wong [1]. An example of particular interest is the differential equation

$$x'' + t^{\lambda} \sin t\, |x|^{\gamma} \operatorname{sgn} x = 0, \tag{1.1}$$

where λ and γ are constants, with $\gamma > 0$. In a series of papers [2–4] Butler showed that, with $\gamma > 1$, the solutions of (1.1) are oscillatory for $\lambda \geqslant -1$. However, with $0 < \gamma < 1$, i.e., in the sublinear case, he succeeded only in proving the oscillation of the equation for $\lambda \geqslant 1$. He made the conjecture that, with $0 < \gamma < 1$, the equation is oscillatory for $\lambda \geqslant -\gamma$. We are indebted to Dr. James S. W. Wong for drawing our attention to this problem. We learned from him that the conjecture which remained unsettled for quite some time has just been proved by him and M. K. Kwong [7].

In this paper, we consider the question of oscillation for differential equations of the more general form:

$$x''(t) + t^{\lambda} p(t)\, f(x) = 0, \qquad t \in [0, \infty), \tag{1.2}$$

where $p(t) \in C[0, \infty)$ is periodic, with period ω, such that $\int_0^{\omega} p(t)\, dt \geqslant 0$, and is not identically zero, and $f(x)$ is an odd function subject to the conditions

$$x^{-\alpha} f(x) \text{ is nondecreasing in } (0, \infty) \tag{1.2a}$$

$$x^{-\beta} f(x) \text{ is nonincreasing in } (0, \infty) \tag{1.2b}$$

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for some constants α, β, where $0 < \alpha \leqslant \beta < 1$. We shall refer to the above properties of $f(x)$ by writing $f(x) \sim [\alpha, \beta]$. The properties of such functions have been studied in some detail by Chen in an earlier paper [5]. It is known that $f(x)$ is absolutely continuous in any finite interval $(0, N)$ and that

$$0 < \alpha \frac{f(x)}{x} \leqslant f'(x) \leqslant \beta \frac{f(x)}{x}, \qquad x > 0, \tag{1.2c}$$

where $f'(x)$ exists almost everywhere. Furthermore, if $x(t) > 0$, $t \geqslant t_0 > 0$ for some t_0, is a solution of (1.2), then

$$\frac{d}{dt}\{f(x(t))\} = f'(x(t))\, x'(t)$$

almost everywhere, where the left-hand side is taken to be zero whenever $x'(t) = 0$ ([6, Lemma 1], with slight modification of the proof therein).

Equation (1.2) clearly includes (1.1) as a special case. Our main result is stated in the following theorem.

THEOREM. *If $p(t)$ and $f(x)$ satisfy the conditions given immediately after* (1.2), *then the continuable solutions of* (1.2) *are oscillatory for* $0 < \lambda < 1$.

2. Some Lemmas

LEMMA 1. *Let $g(t)$ and $p(t)$ be functions satisfying the following conditions*:

(i) *$g(t)$ is continuous and strictly decreasing to zero on $[0, \infty)$ as $t \to \infty$.*

(ii) *$p(t) \not\equiv 0$ is continuous and periodic on $[0, \infty)$, with period ω and zero integral mean, i.e., $\int_0^\omega p(\tau)\, d\tau = 0$.*

Then there exist t_1 and $t_2 \in [0, \omega]$ such that

(A) $\int_{t_1}^t g(\tau)\, p(\tau)\, d\tau \geqslant \varepsilon_1 > 0$,

for $t \geqslant t_1 + \omega$ and a constant ε_1,

(B) $\int_{t_2}^t g(\tau)\, p(\tau)\, d\tau \leqslant -\varepsilon_2 < 0$,

for $t \geqslant t_2 + \omega$ and a constant ε_2, and

(C) $\left|\int_{t_0}^\infty g(\tau)\, p(\tau)\, d\tau\right| \leqslant K g(t_0)$

for some constant $K > 0$ and all $t_0 \in [0, \infty)$.

Proof. Let $p_1(t) = \int_0^t p(\tau)\,d\tau$. Then $p_1(t)$ is continuous and periodic, with period ω. Hence there exist t_1 and t_2 such that

$$m = p_1(t_1) \leqslant p_1(t) \leqslant p_1(t_2) = M, \qquad \text{for all } t \geqslant 0,$$

where m and M are, respectively, the proper minimum and proper maximum of $p_1(t)$, with $m < M$ since $p(t) \not\equiv 0$.

Also let

$$\tilde{p}_1(t) = \int_{t_1}^{t_1+t} p(\tau)\,d\tau.$$

Then $\tilde{p}_1(t) = \tilde{p}_1(t+\omega)$ and $\tilde{p}_1(t) \geqslant 0$ for $t \geqslant 0$.

To prove (A), we note that

$$\int_{t_1}^{t} g(\tau)\,p(\tau)\,d\tau = \int_0^{t-t_1} g(t_1+\tau)\,d\tilde{p}_1(\tau)$$
$$= g(t)\,\tilde{p}_1(t-t_1) - \int_0^{\omega} \tilde{p}_1(\tau)\,dg(t_1+\tau) - \int_{\omega}^{t-t_1} \tilde{p}_1(\tau)\,dg(t_1+\tau).$$

Since $\tilde{p}_1(\tau) \geqslant 0$ for $\tau \geqslant 0$ and $g(\tau) \downarrow 0$ (strictly decreasing) and $\tilde{p}_1(\tau) \not\equiv 0$ in $[0, \omega]$, we have

$$\int_{t_1}^{t} g(\tau)\,p(\tau)\,d\tau \geqslant -\int_0^{\omega} \tilde{p}_1(\tau)\,dg(t_1+\tau) = \varepsilon_1 > 0,$$

for $t \geqslant t_1 + \omega$.

For the proof of (B), it is easily seen that (B) is just equivalent to (A) when we replace $p(t)$ by $-p(t)$ and t_1 by t_2.

To prove (C), we make use of the second mean value theorem so that

$$\int_{t_0}^{t} g(\tau)\,p(\tau)\,d\tau = g(t_0)\int_{t_0}^{t'} p(\tau)\,d\tau + g(t)\int_{t'}^{t} p(\tau)\,d\tau$$

for some $t' \in [t_0, t]$.

Hence

$$\int_{t_0}^{t} g(\tau)\,p(\tau)\,d\tau = p_1(t')(g(t_0) - g(t)) + p_1(t)\,g(t) - g(t_0)\,p_1(t_0)$$

and

$$\left|\int_{t_0}^{t} g(\tau)\,p(\tau)\,d\tau\right| \leqslant 3Mg(t_0), \qquad \text{for} \quad t \geqslant t_0.$$

From the above inequality, we conclude that the integral

$$\mathcal{I}(t_0, t) = \int_{t_0}^{t} g(\tau)\, p(\tau)\, d\tau$$

converges and

$$\mathcal{I}(t_0, t) = 0(g(t_0)), \qquad \text{as} \quad t \to \infty.$$

Remark. It is easy to see that results similar to (A) and (B) can also be obtained for t in any finite interval $[0, N]$ if $g(t) \uparrow A \leqslant \infty$ (strictly increasing) as $t \to \infty$.

LEMMA 2. *Consider the functions*

$$\phi(t) = \int_{t_0}^{t} \tau^{\lambda} p(\tau)\, d\tau, \qquad 0 < \lambda < 1,$$

and

$$\phi_1(t) = \int_{t_0}^{t} \phi(\tau)\, d\tau,$$

where $p(t)$ satisfies the conditions of Lemma 1. *Then*

(i) $\phi(t) \geqslant Kt^{\lambda}$ *in "half-measure" for some constant $K > 0$. By this, we mean that on any interval $[\bar{t}, \bar{t} + \omega]$, for $\bar{t}$ sufficiently large, there exists a subset E of the interval, such that $m(E) \geqslant \sigma\omega$ for some σ, $0 < \sigma < 1$, independent of $\bar{t}$.*

(ii) $|\phi_1(t)| \leqslant K't$, *for some constant $K' > 0$ and t sufficiently large.*

Proof. We introduce the following functions:

(a) $p_1(t) = \int_0^t p(\tau)\, d\tau$.

(b) $p_1^*(t) = p_1(t) - c_0$, with constant c_0 chosen so that $p_1^*(t)$ is of zero integral mean.

(c) $p_2^*(t) = \int_0^t p_1^*(\tau)\, d\tau$.

(d) $p_2^{\Delta}(t) = p_2^*(t) - c_1$, with constant c_1 chosen so that $p_2^{\Delta}(t)$ is of zero integral mean.

Then, on integrating by parts, we have

$$\phi(t) = c_2 + t^{\lambda} p_1^*(t) - \lambda \int_{t_0}^{t} \tau^{\lambda - 1} p_1^*(\tau)\, d\tau,$$

where c_2 is a constant depending only on t_0 for given λ and $p(t)$. Since $\tau^{\lambda - 1}$

decreases steadily to zero in $[1, \infty)$, by Lemma 1, there exist t_1 and t_2 such that

$$\int_{t_1}^{t} \tau^{\lambda-1} p_1^*(\tau)\, d\tau > 0 \qquad \text{and} \qquad \int_{t_2}^{t} \tau^{\lambda-1} p_1^*(\tau)\, d\tau < 0$$

for all $t \geqslant \max(t_1 + \omega, t_2 + \omega)$. Hence for such t we have

$$c_3' + t^\lambda p_1^*(t) \leqslant \phi(t) \leqslant c_3 + t^\lambda p_1^*(t), \tag{2.1}$$

where

$$c_3' = c_2 - \lambda \int_{t_0}^{t_2} \tau^{\lambda-1} p_1^*(\tau)\, d\tau$$

and

$$c_3 = c_2 - \lambda \int_{t_0}^{t_1} \tau^{\lambda-1} p_1^*(\tau)\, d\tau.$$

Since $p_1^*(t)$ is periodic, with period ω and zero integral mean, and is not identically zero, there exists on any arbitrary interval $[\tilde{t}, \tilde{t} + \omega]$, $\tilde{t} \in [0, \infty)$, a subset E such that

(i) $p_1^*(t) \geqslant \delta > 0$, $t \in E$,

(ii) $m(E) = \sigma\omega$, $0 < \sigma < 1$,

where δ, σ are constants independent of $\tilde{t}$. Thus for sufficiently large $\tilde{t}$, say, $\tilde{t} \geqslant T$, we have from (2.1)

$$\phi(t) \geqslant c_3' + t^\lambda p_1^*(t) \geqslant \tfrac{1}{2} t^\lambda p_1^*(t) > K t^\lambda,$$

for some constant $K > 0$, whenever $t \in E \subset [\tilde{t}, \tilde{t} + \omega]$.

Next, consider $\phi_1(t)$. On integrating by parts and expressing the functions in terms of $p_1^*(t)$ and $p_2^\Delta(t)$, we have

$$\phi_1(t) = c_4 + c_5 t + (1 + \lambda)\, t^\lambda p_2^\Delta(t) - \lambda(1 + \lambda) \int_{t_0}^{t} \tau^{\lambda-1} p_2^\Delta(\tau)\, d\tau$$
$$- \lambda t \int_{t_0}^{t} \tau^{\lambda-1} p_1^*(\tau)\, d\tau,$$

where c_4 and c_5 are constants. By (C) of Lemma 1, we have

$$\left| \int_{t_0}^{t} \tau^{\lambda-1} p_1^*(\tau)\, d\tau \right| \leqslant K t_0^{\lambda-1} \qquad \text{and} \qquad \left| \int_{t_0}^{t} \tau^{\lambda-1} p_2^\Delta(\tau)\, d\tau \right| \leqslant K t_0^{\lambda-1}$$

for some constant $K > 0$. Also $p_2^{\Delta}(t)$ is bounded for all t. Hence

$$|\phi_1(t)| \leqslant K't$$

for some constant $K' > 0$ and t sufficiently large, say, $t \geqslant T$.

3. Proof of Main Theorem

Proof. Suppose that, on the contrary, there is a continuable solution $x(t)$ of (1.2) which is non-oscillatory. Then, without loss of generality, we may assume that $x(t) > 0$ for t sufficiently large. In terms of the transformation $z = x'/f(x)$, (1.2) can be written as

$$z'(t) = -t^{\lambda} p(t) - f'(x(t))\, z^2(t). \tag{3.1}$$

Consider first the proof of the theorem for $p(t)$ with zero integral mean.

Integrating (3.1), we have

$$z(t) = z(t_0) - \phi(t) - \mathscr{U}(t), \tag{3.2}$$

where

$$\phi(t) = \int_{t_0}^{t} \tau^{\lambda} p(\tau)\, d\tau$$

$$\mathscr{U}(t) = \int_{t_0}^{t} f'(x(\tau))\, z^2(\tau)\, d\tau.$$

Again integrating (3.2), we have

$$\int_{t_0}^{t} z(\tau)\, d\tau = z(t_0)(t - t_0) - \phi_1(t) - \mathscr{U}_1(t), \tag{3.3}$$

where

$$\phi_1(t) = \int_{t_0}^{t} \phi(\tau)\, d\tau \qquad \text{and} \qquad \mathscr{U}_1(t) = \int_{t_0}^{t} \mathscr{U}(\tau)\, d\tau.$$

In view of (1.2c), we have two possible cases:

Case I. $\mathscr{U}(t) \uparrow \infty$ as $t \to \infty$. Then, for t sufficiently large, we have

$$\mathscr{U}_1(t) > Nt$$

for any arbitrarily large positive constant N. By Lemma 2, $|\phi_1(t)| \leqslant Kt$, for t sufficiently large. Hence (3.3) implies that

$$\int_{t_0}^{t} z(\tau)\,d\tau = \int_{x(t_0)}^{x(t)} \frac{du}{f(u)} \leqslant -\bar{N}t, \qquad \text{as} \quad t \to \infty, \tag{3.4}$$

where $\bar{N}$ can be any arbitrarily large positive constant. However,

$$\int_{x_0}^{x} \frac{du}{f(u)} = \frac{x}{f(x)} - \frac{x_0}{f(x_0)} + \int_{x_0}^{x} \frac{uf'(u)}{f^2(u)}\,du$$

so that using (1.2c) we have

$$\frac{1}{(1-\alpha)}\left(\frac{x}{f(x)} - \frac{x_0}{f(x_0)}\right) \leqslant \int_{x_0}^{x} \frac{du}{f(u)} \leqslant \frac{1}{(1-\beta)}\left(\frac{x}{f(x)} - \frac{x_0}{f(x_0)}\right). \tag{3.5}$$

Hence it follows from (3.4) that

$$\frac{1}{(1-\alpha)}\left(\frac{x}{f(x)} - \frac{x_0}{f(x_0)}\right) \leqslant -\bar{N}t, \qquad \text{as} \quad t \to \infty.$$

By assumption, $x(t) > 0$ and $f(x) > 0$, and we have a contradiction.

Case II. $\mathscr{V}(t) \uparrow A < \infty$ for $A > 0$ as $t \to \infty$. From (3.3) and Lemma 2, again we have

$$\int_{t_0}^{t} z(\tau)\,d\tau = 0(t), \qquad \text{as} \quad t \to \infty.$$

Therefore it follows from (3.5) that

$$\frac{x}{f(x)} \leqslant K_1 t \qquad \text{and} \qquad \frac{f(x)}{x} \geqslant K_1' t^{-1}$$

for some constants $K_1, K_1' > 0$ and t sufficiently large. On the other hand from (3.2) and Lemma 2, we have

$$|z(t)| \geqslant K_2 t^{\lambda} \qquad \text{in ``half-measure''}$$

for $K_2 > 0$ and t sufficiently large, say, $t \geqslant T$. Hence

$$\begin{aligned}\mathscr{V}(t) &= \int_{t_0}^{t} f'(x(\tau))\,z^2(\tau)\,d\tau \\ &\geqslant \alpha \int_{t_0}^{t} \frac{f(x(\tau))}{x(\tau)}\,z^2(\tau)\,d\tau \\ &\geqslant K_3 \int_{t_0}^{t} \tau^{2\lambda-1}\,d\tau, \qquad t \geqslant T,\end{aligned}$$

for some constant $K_3 > 0$. Therefore $\mathscr{U}(t) \uparrow \infty$ as $t \to \infty$ and we have a contradiction.

If $(1/\omega) \int_0^\omega p(\tau)\, d\tau = c > 0$, define $p^*(t)$ by $p^*(t) = p(t) - c$ so that $p^*(t)$ is of zero integral mean. Equation (3.2) now becomes

$$z(t) = z(t_0) - \frac{c}{1+\lambda}\,(t^{\lambda+1} - t_0^{\lambda+1}) - \phi(t) - \mathscr{U}(t), \tag{3.6}$$

where $\phi(t)$, now defined with respect to $p^*(t)$, satisfies (2.1). Hence

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \frac{x'(t)}{f(x(t))} = -\infty.$$

By an argument similar to that for Case I, we again arrive at a contradiction. Hence the theorem is proved.

Remark. When $p(t)$ is of zero integral mean, our result is actually true also for an even function $f(x)$, because in that case Eq. (1.2) remains unchanged when $x(t)$ and $p(t)$ are replaced by $-x(t)$ and $-p(t)$ simultaneously so that in the proof we can still assume, without loss of generality, that $x(t) > 0$ for t sufficiently large, and the proof remains valid.

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